JOURNAL OF APPROXIMATION THEORY 1, 412–419 (1968)

On Mean Approximation of Holomorphic Functions

DAVID BELL

Department of Mathematics, Rice University, Houston, Texas 77001

This paper is concerned with extensions of results found in [3]. We shall consider the problem of approximating by certain rational functions, elements of the Banach space $A_{r,p}(V)$ with norm $\| \|_{V,r,p}$ corresponding to a bounded open subset V of the complex plane $\mathbf{C}$ and a pair of real numbers $r \leqslant 0$ and $1 \leqslant p \leqslant \infty$.

$$A_{r,p}(V) = \{f\colon f\colon V \to \mathbf{C} \text{ is holomorphic and } \|f\|_{V,r,p} < \infty\},$$

$$\|f\|_{V,r,p} = \left[\int\!\!\int_V K_V{}^r(\zeta,\zeta)|f(\zeta)|^p \, d(\zeta)\right]^{1/p} \qquad \text{for } 1 \leqslant p < \infty,$$

$$\|f\|_{V,r,\infty} = \sup\{K_V{}^r(\zeta,\zeta)|f(\zeta)|: \zeta \in V\}.$$

Here $K_V(z,\zeta)$ denotes the Bergman kernel function of V, and $d(\zeta)$ denotes integration in ζ in planar measure.

DEFINITION. Let X and Y be components of $\mathbf{C} - \operatorname{clos} V$ where $\operatorname{clos} S$ means the closure of the set S in $\mathbf{C}$. We use the notation $X \leqslant Y$ to mean that there exists a sequence $W_1 = X, W_2, \ldots, W_n = Y$ of (not necessarily distinct) components of $\mathbf{C} - \operatorname{clos} V$ with the following property: For each $j = 2, 3, \ldots, n$ there exists a simple closed curve $\alpha_j\colon [0, t_j] \to \operatorname{clos} W_j$; such that

a. α_j is twice continuously differentiable as a function of arc length t.

b. The bounded component of $\mathbf{C} - \{\alpha_j(t)\colon 0 \leqslant t \leqslant t_j\}$ is contained in W_j.

c. $\int_0^{t_j} |\log \delta_j \circ \alpha_j(t)| \, dt = \infty$, where $\delta_j(z) = \inf\{|z - w|\colon w \in W_{j-1}\}$.

The relation $\leqslant$ will then be symmetric and transitive.

THEOREM. *Let V be a bounded open subset of $\mathbf{C}$ such that*

i. $\operatorname{clos}(\mathbf{C} - \operatorname{clos} V) = \mathbf{C} - V$, *and*

ii. *there exist constants $c_1 > 0$ and $c_2 < \infty$ such that $c_1 \leqslant K_V(z,z)\delta_V{}^2(z) \leqslant c_2$ for all $z \in V$, where $\delta_V(z) = \inf\{|z - w|\colon w \in \mathbf{C} - V\}$. (The latter condition will automatically hold if each component of V is simply connected.) If $2r$ is not an integer, assume that each component of V is simply connected.*

Let M be a linear subspace of $A_{r,p}(V)$ such that

iii. *given a component Y of $\mathbf{C} - \operatorname{clos} V$, either $W_0 =$ (the unbounded component of $\mathbf{C} - \operatorname{clos} V) \leqslant Y$, or there exists a component X of $\mathbf{C} - \operatorname{clos} V$ such that*

$X \leqslant Y$ and M contains every rational function all of whose poles are simple and lie in ∂X (the boundary of X). (If ∂X is a finite collection of disjoint Jordan curves, we need only assume that M contains all rational functions all of whose poles are simple and lie in some fixed subarc of ∂X.)

If $r \leqslant 0$ and $p = 1$, or if $r \leqslant -1$ and $1 < p < 2$, then M is dense in $A_{r,p}(V)$.

Remarks. (i) Without knowing anything about the relationship between the various components of $\mathbf{C} - \operatorname{clos} V$, one sees that letting M consist of all rational functions all of whose poles are simple and lie in $(\mathbf{C} - \operatorname{clos} W_0) \cap \partial V$, one can always assert the above density properties of M.

(ii) In the event that $W_0 \leqslant W$ for every component W of $\mathbf{C} - \operatorname{clos} V$, one can take M to be the set $\mathbf{C}[z]$ of all polynomials in z.

PROPOSITION 1. *Let V be a bounded open subset of $\mathbf{C}$ such that $\mathbf{C} - V$ has only finitely many components, and each has interior points. Then for $r \leqslant 0$, $A_{r,1}(V) \subset A_{r-1,\infty}(V)$ and the inclusion map is continuous.*

COROLLARY 1. *If M, V, r and p satisfy the hypotheses* i, ii, *and* iii *of the Theorem, and the hypotheses of Proposition* 1, *then every L_p $(1 \leqslant p \leqslant \infty)$ analytic function on V is the limit under the norm $\| \ \|_{V,-1,\infty}$ of a suitable sequence of elements of M.*

We next see that hypothesis iii of the Theorem is in certain circumstances a condition which is necessary for M to be dense in $A_{r,p}(V)$.

PROPOSITION 2. *Let α, β and γ be three simple closed curves such that γ lies inside β, and β lies inside α. We allow the possibility that α and β, or β and γ have points in common. Assume that β: $[0,L] \to \mathbf{C}$ is twice continuously differentiable as a function of arc length. Let D be the open set consisting of all points strictly inside α and strictly outside γ. Suppose that $\int_0^L \log \delta_D \circ \beta(t)\, dt > -\infty$. Then for any constant $c > 0$ and any $r \leqslant 0$, the polynomials belonging to $\{f: \|f\|_{D,r,1} \leqslant c\}$ form a normal family on the set of points inside α.*

COROLLARY 2. *Let V be an open subset of $\mathbf{C}$, M some collection of rational functions all of whose poles are simple and lie in $\mathbf{C} - V$. Let α, β, and γ be as in Proposition 2, and assume that the set D described in Proposition 2 is contained in V. If there exists a limit point of $\mathbf{C} - V$ inside β, and if the set of all points z lying inside α such that at least one element of M has a pole at z, is finite, then for $r \leqslant 0$ and $1 \leqslant p < 2$, M is not dense in $A_{r,p}(V)$.*

Example. Consider the situation depicted in Figure 1. Assume that α and β are simple closed Jordan curves, tangent to the real axis at each intersection point,

$$\int_{-1/2}^{1/2} \log \delta_V(t)\, dt > -\infty, \qquad \text{and} \qquad \int_0^L \log \delta_{\mathbf{C}-W_0} \circ \alpha(t)\, dt = -\infty,$$

where α: $[0,L] \to \mathbf{C}$ is parametrized by arc length and is twice continuously differentiable, and $W_0 = \{z: |z| > 1\}$. Then the set of all rational functions all of whose poles are simple and lie inside (or on) β is dense in $A_{r,p}(V)$ if $r \leqslant 0$ and $p = 0$, or if $r \leqslant -1$ and $1 < p < 2$. Conversely, if $r \leqslant 0$ and $1 \leqslant p \leqslant \infty$, if M is some collection of rational functions all of whose poles are simple and lie in $\mathbf{C} - V$, and if M is dense in $A_{r,p}(V)$, then there exist infinitely many points z inside or on β for which one can find an element of M having a pole at z.

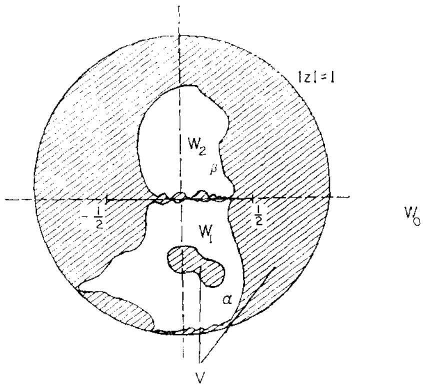

FIG. 1.

The existence of constants $c_1 > 0$, $c_2 < \infty$ *such that* $c_1 \leqslant K_V(\zeta, \zeta)\delta_V^2(\zeta) \leqslant c_2$ for *all* $\zeta \in V$: To see that this condition holds only for special V, note that $K_V = K_{V'}$ whenever V' is obtained from V by removing finitely many points. On the other hand, we have: *Suppose V satisfies the hypotheses of Proposition* 1. *Let* $S_0, \ldots, S_n$ *be the components of* $\mathbf{C} - V$, *where* S_0 *is unbounded, and assume that there exists an interior point* p_j *of* S_j, *for each* $j = 1, 2, \ldots, n$. *Then there exist constants* $c_1 > 0$ *and* $c_2 < \infty$ *(perhaps depending on V) such that* $c_1 \leqslant K_V(\zeta, \zeta)\delta_V^2(\zeta) \leqslant c_2$ *for all* $\zeta \in V$.

Proof. (i) Choose $c_1 > 0$ and $c_2 < \infty$ such that $c_1 \leqslant K_D(z,z)\delta_D^2(z) \leqslant c_2$ for every simply-connected domain D in $\mathbf{C}$, and every z in D. Given $z_0 \in \partial S_0 \cap \partial V$, one can find a neighborhood W of z_0 such that every component of $E = W \cap V$ is simply connected. On the other hand, $V \subset F = V \cup S_1 \cup \ldots \cup S_n$, where each component of F is simply-connected. Choose a neighborhood H of z_0 such that for $z \in H \cap V$, $\delta_E(z) = \delta_V(z) = \delta_F(z)$. Then for $z \in H \cap V$,

$$c_1 \leqslant K_F(z,z)\,\delta_F{}^2(z) \leqslant K_V(z,z)\,\delta_V{}^2(z) \leqslant K_E(z,z)\,\delta_E{}^2(z) \leqslant c_2.$$

So $c_1 \leqslant K_V(z,z)\delta_V{}^2(z) \leqslant c_2$ holds for $z \in V \cap W$, for some neighborhood W (generally, one different from the above) of $\partial S_0 \cap \partial V$.

(ii) Next, consider $z_0 \in S_j \cap \partial V$, $1 \leqslant j \leqslant n$, and let $0 < \epsilon < 1$. Define $T(z) = 1/(z - p_j)$ for $z \in \mathbf{C} - \{p_j\}$. Then the bounded open set $T(V)$ satisfies the hypotheses of Proposition 1, and $T(z_0)$ is in the unbounded component of $\mathbf{C} - T(V)$, so by (i), one can choose a neighborhood W of z_0 such that if $z \in W \cap V$, then

$$c_1 \leqslant K_{T(V)}(T(z), T(z))\,\delta^2_{T(V)}(T(z)) \leqslant c_2.$$

Furthermore, by choosing W sufficiently fine, one can also easily show that for $z \in W \cap V$,

$$|T'(z)|(1 - \epsilon) \leqslant \delta_{T(V)}(T(z))/\delta_V(z) \leqslant |T'(z)|(1 + \epsilon).$$

It then follows from the transformation formula for the Bergman kernel function, that for $z \in W \cap V$,

$$(1 + \epsilon)^{-2} c_1 \leqslant K_V(z,z)\,\delta_V{}^2(z) \leqslant (1 - \epsilon)^{-2} c_2.$$

By the compactness of ∂V, this also holds for $z \in W \cap V$, where W is some open set containing ∂V.

Hence, if V satisfies the hypotheses of Proposition 1, then there exist constants $c_1 > 0$ and $c_2 < \infty$ (which may depend on V), such that $c_1 \leqslant K_V(z,z)\delta_V{}^2(z) \leqslant c_2$ for all $z \in V$.

Proof of Proposition 1. Let $z_0 \in \partial V$, where V satisfies the hypotheses of Proposition 1. Choose a neighborhood N of z_0, such that every component of $N \cap V$ is simply connected. As shown in [4], page 199, one can find a constant c such that for any bounded simply connected domain D, any $r \leqslant 0$, and any $f \in A_{r,1}(D)$, $\|f\|_{D,r-1,\infty} \leqslant c\|f\|_{D,r,1}$. So

$$\|f\|_{V,r,1} \geqslant \|f\|_{N \cap V,r,1} \geqslant c\|f\|_{N \cap V,r-1,\infty} \geqslant c'|\delta^{2-2r}_{N \cap V}(z) f(z)|$$

for $z \in N \cap V$. Then for some constant c'' and a sufficiently fine neighborhood H of z_0, $\|f\|_{V,r,1} \geqslant c'' K_V^{r-1}(z,z)|f(z)|$ for $z \in H \cap V$. It is also easy to show that similar estimates hold on compact subsets of V, concluding the proof of Proposition 1.

Proof of Corollary 1. For $1 \leqslant p \leqslant \infty$, we have the continuous inclusions

$$A_{0,p}(V) \subset A_{0,1}(V) \subset A_{-1,\infty}(V).$$

Proof of Proposition 2. There exists a constant c, such that for any $f \in A_{r,1}(V)$ and $z \in V$, $|f(z)| \leqslant c\delta_V^{2r-2}(z)\|f\|_{r,1,V}$. The result then follows from the results in [6] on the differentiability of a conformal mapping T of the unit disk onto the set of points inside β, and the application of the Poisson–Jensen formula to the functions $f \circ T$. (If f has a zero of order $n > 0$ at $T(0)$, we, instead, consider $f \circ T(z)/z^n$, whose modulus is greater than or equal to that of $f \circ T$ for $|z| \leqslant 1$, with equality for $|z| = 1$.)

Proof of the Theorem. Let $1 \leqslant p < 2$, and let $l\colon A_{r,p}(V) \to \mathbf{C}$ be a continuous linear functional, such that $l(M) = \{0\}$. It will be shown that $l(A_{r,p}(V)) = \{0\}$. Obtain a function $\lambda \in L_q(\mathbf{C})$ (where $1/p + 1/q = 1$), such that $\lambda(z) = 0$ for $z \in \mathbf{C} - V$, and

$$l(f) = \int_V \lambda(\zeta)\, K_V{}^r(\zeta, \zeta) f(\zeta)\, d(\zeta)$$

for each $f \in A_{r,p}(V)$. We shall construct a function $h\colon \mathbf{C} \to \mathbf{C}$, such that

1. h has generalized partial derivatives which satisfy $\partial/\partial\bar{z}h(z) = 0$ if $z \in \mathbf{C} - V$, and $\partial/\partial\bar{z}h(z) = K_V{}^r(z,z)\lambda(z)$ almost everywhere on V.
2. h is continuous on $\mathbf{C}$, and $h(z) = 0$ if $z \in \mathbf{C} - V$.
3. If $p = 1$, then for all $z \in V$,

$$|h(z)| \leqslant c_3 K_V{}^r(z,z)\, \delta_V(z)\, [1 + \log^+(1/\delta_V(z))]$$

for some constant c_3 which does not depend on z. Here $\log^+(t)$ means $\max\{0, \log t\}$.

4. If $1 < p < 2$, then for $z \in V$,

$$|h(z)| \leqslant c_p K_V{}^r(z,z)\, \delta_V^{1-2/q}(z),$$

where c_p is a constant which does not depend on z.

We then construct the function

$$\omega_n(z) = j([\log\log(\rho/\delta_V(z))]/n),$$

where $j\colon R \to R$ is continuously differentiable (R = the real numbers), $j(t) = 1$ for $t \leqslant 0$, $j(t) = 0$ for $t \geqslant 1$, and $0 \leqslant j(t) \leqslant 1$ for $0 < t < 1$, and $\rho = 1 +$ the diameter of V. By the boundedness of $j'(t)$ and the fact that δ_V is Lipschitzian, we can find a constant c_4 such that for all $z \in V$,

5. $|\partial/\partial\bar{z}\omega_n(z)| \leqslant c_4/[n\delta_V(z)\log(\rho/\delta_V(z))]$.

Here c_4 does not depend on n. Here, too, differentiation is in the generalized sense.

One next shows that, for $f \in A_{r,p}(V)$,

6. $\int_V \omega_n(\zeta) f(\zeta)\, K_V{}^r(\zeta, \zeta)\, \lambda(\zeta)\, d(\zeta) = -\int_V (\partial/\partial\bar{\zeta}\omega_n(\zeta)) f(\zeta)\, h(\zeta)\, d(\zeta)$.

We first prove 6. Let n be fixed; then choose a compact set L in V, and a sequence g_k of functions on $\mathbf{C}$ into $\mathbf{C}$, with continuous first partial derivatives with respect to x and to y on $\mathbf{C}$, such that L contains the support of ω_n and of each g_k, and

$$\lim_{k\to\infty} \|\omega_n - g_k\|_{V,0,p} = 0 = \lim_{k\to\infty} \|\partial/\partial\bar{z}\omega_n - \partial/\partial\bar{z}g_k\|_{V,0,p}.$$

Furthermore, for each k,

$$\int_V g_k(\zeta) f(\zeta) K_V{}^r(\zeta,\zeta) \lambda(\zeta)\, d(\zeta) = -\int_V (\partial/\partial\bar{\zeta} g_k(\zeta))\, h(\zeta)\, d(\zeta),$$

by 1. Hence 6 follows by taking the limit as k approaches ∞.

If $p = 1$, using 3, 5, and 6, one sees that

$$|l(f)|$$
$$\leqslant \lim_{n\to\infty} \int_V |f(\zeta)| c_3 K_V{}^r(\zeta,\zeta)\, \delta_V(\zeta)[1 + \log^+(1/\delta_V(\zeta))]\, c_4/[n\delta_V(\zeta)\log(\rho/\delta_V(\zeta)]\, d(\zeta)$$
$$\leqslant \lim_{n\to\infty} (1/n)\, c_3 c_4 [1 + 1/\log\rho] \|f\|_{V,r,1} = 0,$$

proving the theorem for $p = 1$.

If $1 < p < 2$, using hypothesis ii and 4, 5, and 6, one sees that

$$\left|\int_\Delta \omega_n(\zeta) f(\zeta) K_V{}^r(\zeta,\zeta) \lambda(\zeta)\, d(\zeta)\right|$$
$$\leqslant (1/n) \int_\Delta |f(\zeta)| c_p K^{r/p}(\zeta,\zeta)\, c_1^{r(1-1/p)}\, \delta_V{}^s(\zeta)\, c_4/\log(\rho/\delta_V(\zeta))\, d(\zeta),$$

where $s = -2(r - r/p) - 2/q = -2(r+1)/q$. Then $l(f) = 0$ if $r \leqslant -1$.

Proof of 1–4. For $z \in \mathbf{C}$ let

$$h(z) = (-1/\pi) \int_V \lambda(\zeta) K_V{}^r(\zeta,\zeta)[1/(\zeta - z)]\, d(\zeta).$$

By [*1*], page 7, and [*2*], page 386, we see that h is continuous on $\mathbf{C}$, holomorphic on the interior of $\mathbf{C} - V$, satisfies 1, and for all z and w in $\mathbf{C}$,

$$|h(z) - h(w)| \leqslant c_5 |z - w|^\epsilon$$

for some positive constants c_5 and ϵ which are independent of z and w. We shall first prove that $h(z) = 0$ for $z \in \mathbf{C} - V$. Let W_1 be a component of $\mathbf{C} - \operatorname{clos} V$. If W_1 is unbounded, using the fact that M contains all polynomials in z and Mergelyan's Theorem, one sees that $h(z) = 0$ for $z \in W_1$. If W_1 is bounded and if M contains all rational functions whose poles are simple and lie in ∂W_1, then $h(z) = 0$ for $z \in \partial W_1$, so $h(z) = 0$ for $z \in W_1$, by the maximum principle.

Now let W_1 and W_2 be two components of $\mathbf{C} - \operatorname{clos} V$, such that $h(z) = 0$ for $z \in W_1$, and such that there exists a simple closed curve $\alpha_2\colon [0, t_2] \to \mathbf{C}$ which is twice continuously differentiable as a function of arc length t, and for which the bounded component $W_2{}'$ of $\mathbf{C} - \{\alpha_2(t)\colon 0 \leqslant t \leqslant t_2\}$ is contained in W_2, and $\int_0^{t_2} |\log \delta_2 \circ \alpha_2(t)| \, dt = \infty$, where $\delta_2(z) = \inf\{|z - w| : w \in W_1\}$. Clearly, $|h(z)| \leq c_5 \delta_2^{\epsilon}(z)$ for all $z \in \mathbf{C}$. To see that $h(z) = 0$ for $z \in W_2$, pick a continuously differentiable homeomorphism T of $\operatorname{clos} \Delta_0$ onto $W_2' \cup \{\alpha_2(t): 0 \leq t \leq t_2\}$, where $\Delta_0 = \{z: |z| < 1\}$, which is holomorphic on Δ_0. (This is possible, by [*6*].) Then

$$\infty = \int_0^{t_2} |\log(\delta_2 \circ \alpha_2(t))| \, dt = \int_0^{2\pi} |\log(\delta_2 \circ T)(e^{i\theta}) \, d/d\theta T(e^{i\theta})| \, d\theta$$

$$\leq \sup\{|d/d\theta T(e^{i\theta})| : 0 \leq \theta \leq 2\pi\} \int_0^{2\pi} |\log \delta_2 \circ T(e^{i\theta})| \, d\theta,$$

implying $\int_0^{2\pi} |h \circ T(e^{i\theta})| \, d\theta = \infty$, so $h(z) = 0$ for $z \in W_2$, by [*5*], page 73. Using this technique, we can prove inductively, that if X and Y are components of $\mathbf{C} - \operatorname{clos} V$, $X \leq Y$, and $h(z) = 0$ for $z \in X$, then $h(z) = 0$ for $z \in Y$. Then, by hypothesis, $h(z) = 0$ for $z \in \operatorname{clos}(\mathbf{C} - \operatorname{clos} V) = \mathbf{C} - V$.

By [*1*], page 7, and hypothesis ii, we see that 3 holds for $r = 0$, so the theorem is true for $r = 0$ and $p = 1$.

We shall now show that h satisfies 3 for all $r < 0$. Choose an integer $m \geq 0$, and a number a, $0 \leq a < 1$, such that $-2r = m + a$. Let $p \in \partial V$, let $f(z)$ be some holomorphic branch of $(z - p)^a$ defined on V, and let $w \in \mathbf{C} - V$. Define

$$H(w,z) = (-1/\pi) \int_V \lambda(\zeta) K_V^r(\zeta, \zeta) / [(\zeta - z)(\zeta - w)^m f(\zeta)] \, d(\zeta).$$

To see that $H(w,z) = 0$ for $z \in \mathbf{C} - V$, first pick $m + 1$ distinct points $z, w_1, \ldots, w_m$ in $\mathbf{C} - \operatorname{clos} V$, and let

$$g(\zeta) = 1/[(\zeta - z)(\zeta - w_1)(\zeta - w_2) \ldots (\zeta - w_m) f(\zeta)].$$

Then $g \in A_{0,1}(V)$, so using our theorem as proven above for $r = 0$, there exists a sequence $R_1, R_2, R_3, \ldots$ of elements of M, such that $\lim_{k \to \infty} \|R_k - g\|_{V,0,1} = 0$. Then

$$0 = \lim_{k \to \infty} \int_V \lambda(\zeta) K_V^r(\zeta, \zeta) R_k(\zeta) \, d(\zeta) = \int_V \lambda(\zeta) K_V^r(\zeta, \zeta) g(\zeta) \, d(\zeta).$$

Using hypothesis ii and [*2*], page 386, one sees that

$$\int_V \lambda(\zeta) K_V^r(\zeta, \zeta) / [(\zeta - z)(\zeta - w_1) \ldots (\zeta - w_n) f(\zeta)] \, d(\zeta)$$

is a continuous function of any one of $z \in \mathbf{C}$ and $w_1, w_2, \ldots, w_m \in \mathbf{C} - V$, with the remaining points fixed. So $H(w,z)$ is continuous in $z \in \mathbf{C}$, and $H(w,z) = 0$ if $z \in \mathbf{C} - V$. Now define $h(w,z)$: $h(w,z) = 0$ if $z \in \mathbf{C} - V$, $h(w,z) = (z - w)^m f(z) H(w,z)$ for $z \in V$. Clearly $h(w,z)$ is continuous in $z \in \mathbf{C}$. Consider $h(w,z) - h(z)$. It is continuous on $\mathbf{C}$, zero on $\mathbf{C} - V$, and since $\partial/\partial\bar{z}[h(w,z) - h(z)] = 0$ almost everywhere on V, $h(w,z) - h(z)$ is holomorphic on V. The maximum principle then implies $h(w,z) = h(z)$ on $\mathbf{C}$.

Then [1], page 7, and hypothesis ii imply that h satisfies 3, completing the proof of the theorem, for $p = 1$.

To prove the Theorem for $1 < p < 2$, we modify the above as follows: If $r = -1$ and $1 < p < 2$, g will be of the form $1/[(\zeta - z)(\zeta - w_1)(\zeta - w_2)]$, so by considering g as a linear combination of $1/(\zeta - z)$, $1/(\zeta - w_1)$, $1/(\zeta - w_2)$ for z, w_1, and w_2 distinct, we see that $H(w, z) = 0$ for $z \in \mathbf{C} - V$. This implies that the Theorem holds for $1 < p < 2$ and $r = -1$. The case $r < -1$ is handled as above, by approximating g by elements of M.

References

1. L. Bers, Theory of pseudo-analytic functions. Mimeographed notes, New York University, Institute for Mathematics and Mechanics, 1953.
2. L. Bers, Riemann's mapping theorem for variable metrics. *Ann. Math.* **72** (1960), 385–404.
3. L. Bers, An approximation theorem. *J. Analyse Math.* **14** (1965), 1–4.
4. L. Bers, Automorphic forms and Poincaré series for infinitely generated Fuchsian groups. *Am. J. Math.* **87** (1965), 196–214.
5. K. Hoffman, "Banach Spaces of Analytic Functions." Prentice-Hall, Inc., Englewood Cliffs, 1962.
6. S. Warschawski, On differentiability at the boundary in conformal mapping. *Proc. Am. Math. Soc.* **12** (1961), 614–620.